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Generalized vertex algebras generated by parafermion-like vertex operators

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Abstract It is proved that for a vector space W, any set of parafermion-like vertex operators on W in a certain canonical way generates a generalized vertex algebra in the sense of [DL2] with W as a natural module. This result generalizes a result of [Li2]. As an application, generalized vertex algebras are constructed from Lepowsky-Wilson's Z-algebras of any nonzero level.

1 Introduction

Vertex operator algebras, introduced in mathematics ([B], [FLM]), are known essentially to be chiral algebras, introduced in physics ([BPZ], [MS]). In terms of physical language, chiral algebras for bosonic field theories are vertex operator algebras while chiral algebras are vertex operator superalgebras for fermionic field theories. In physics, further generalizations of bosons and fermions are parafermions ([ZF1-2], [G]), where the chiral algebras were called parafermion algebras. Independently (and earlier), Z-operators and Z-algebras ([LW1-6], [LP1-2]) were introduced in mathematics to study standard modules for affine Lie algebras. In [DL1-3] and [LP1-2], the relations between Z-operators and parafermion operators were clarified. Furthermore, in [DL2], Lepowsky-Wilson's Z-algebras were put into larger, more natural algebras where the notions of generalized vertex (operator) algebra and abelian intertwining algebras were introduced in [DL2]. Generalized algebraic structures associated with rational lattices were also studied in [M] and a notion called vertex operator para-algebra was independently introduced and studied in [FFR].

Roughly speaking, parafermion algebras are generalized vertex operator algebras. For bosonic or fermionic field operators $a(z)$ and $b(z)$, the locality amounts to

$$(z_1 - z_2)^k a(z_1)b(z_2) = (-1)^{|a||b|}(z_1 - z_2)^k b(z_2)a(z_1) \tag{1.1}$$

for some non-negative integer k, where $|a| = 0$ if $a(z)$ is bosonic and 1 if $a(z)$ is fermionic. Parafermions are always associated to an abelian group G equipped with a $\mathbf{C}^{\times}$-valued alternating $\mathbf{Z}$-bilinear form $c(\cdot, \cdot)$ and a $\mathbf{C}/2\mathbf{Z}$-valued $\mathbf{Z}$-bilinear form $(\cdot, \cdot)$ on G. For

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parafermion operators $a(z), b(z)$ with gradings $g, h \in G$, the following relation holds:

$$(z_1 - z_2)^{k+(g,h)} a(z_1) b(z_2) = (-1)^k c(g,h) (z_2 - z_1)^{k+(g,h)} b(z_2) a(z_1) \qquad (1.2)$$

for some nonnegative integer k.

In physical literatures, a chiral algebra is often described by a set of generating field operators and a certain set of relations (such as operator product expansions). Now we know that a certain set of field operators on a vector space indeed gives rise to a vertex operator (super)algebra. A result proved in [Li2] is that any set of mutually local vertex operators on a vector space W generates a canonical vertex superalgebra with W as a module. (This result was extended in [Li3] for twisted modules.) This is an analogue of the simple fact in linear algebra that any set of mutually commutative endomorphisms on a vector space U generates a commutative associative algebra with U as a module. (See [FKRW], [LZ], [MN], [MP] and [X] for other related interesting results.)

As the main result of this paper, we extend the results of [Li2-3] for the notion of generalized vertex algebra [DL2]. This paper is modeled on [Li2], however, the two key theorems (Propositions 3.8 and 3.13) require essentially new proofs.

It seems that the most general and natural notion is the one of abelian intertwining algebra [DL2]. An abelian intertwining algebra by definition is associated to an abelian group G and $\mathbf{C}^{\times}$-valued functions F on $G \times G \times G$ and Ω on $G \times G$ satisfying certain conditions. It would be nice to extend our result to the notion of abelian intertwining algebra. When we were trying, we found that the extension is almost straightforward except for extending the two key theorems (Propositions 3.8 and 3.13) we need certain identities purely about F and Ω. We can prove that one of the identities follows from the assumptions on F and Ω, but we are not be able to prove the others. We hope to discuss this issue in some other place.

This paper consists of four sections including this introduction as the first section. In Section 2, we recall some basic definitions and results from [DL2]. The main result of this paper is given in Section 3. In Section 4, we construct canonical generalized vertex algebras from Z-algebras of any nonzero level.

2 Definition and duality for generalized vertex algebras

This section is preliminary. In this section, we recall from [DL2] the basic definitions (of generalized vertex algebra and module) and basic duality properties.

First, let us briefly review some formal variable calculus. (Best references are [FLM] and [FHL].) Throughout this paper, z, z_0, z_1, z_2 and x, y will be mutually commuting (independent) formal variables. We shall use $\mathbb{N}$ for the set of all nonnegative integers, $\mathbf{Z}_+$

for the set of positive integers and $\mathbf{C}$ for the set of complex numbers. All vector spaces are assumed to be over $\mathbf{C}$.

For a vector space U, set

$$U\{z\} = \left\{ \sum_{n\in\mathbf{C}} u(n)z^n \mid u(n) \in U \quad \text{for } n \in \mathbf{C} \right\}. \tag{2.1}$$

Let $D = d/dz$ be the formal differential operator on $U\{z\}$:

$$D\left(\sum_{n\in\mathbf{C}} u(n)z^n\right) = \sum_{n\in\mathbf{C}} nu(n)z^{n-1}. \tag{2.2}$$

The formal residue operator Res_z from $U\{z\}$ to U is defined by

$$\mathrm{Res}_z u(z) = u(-1) \tag{2.3}$$

for $u(z) = \sum_{n\in\mathbf{C}} u(n)z^n \in U\{z\}$.

The following are useful subspaces of $U\{z\}$:

$$\begin{aligned} U[[z, z^{-1}]] &= \left\{ \sum_{n\in\mathbf{Z}} u(n)z^n \mid u(n) \in U \quad \text{for } n \in \mathbf{Z} \right\}, & (2.4)\\ U((z)) &= \left\{ \sum_{n\in\mathbf{Z}} u(n)z^n \in U[[z, z^{-1}]] \mid u(n) = 0 \quad \text{for } n \text{ sufficiently small} \right\}, & (2.5)\\ U[[z]] &= \left\{ \sum_{n\in\mathbf{Z}} u(n)z^n \in U[[z, z^{-1}]] \mid u(n) = 0 \quad \text{for } n < 0 \right\}. & (2.6) \end{aligned}$$

A typical element of $\mathbf{C}[[\mathbf{z}, \mathbf{z}^{-1}]]$ is the formal Fourier expansion of the delta-function at 0:

$$\delta(z) = \sum_{n\in\mathbf{Z}} z^n. \tag{2.7}$$

Its fundamental property is:

$$f(z)\delta(z) = f(1)\delta(z) \quad \text{for } \; f(z) \in \mathbf{C}[\mathbf{z}, \mathbf{z}^{-1}]. \tag{2.8}$$

For $\alpha \in \mathbf{C}$, by definition,

$$(z_1 - z_2)^\alpha = \sum_{i\geq 0} \binom{\alpha}{i} (-1)^i z_1^{\alpha-i} z_2^i. \tag{2.9}$$

Then

$$\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbf{Z}} \left(\frac{z_1 - z_2}{z_0}\right)^n = \sum_{n\in\mathbf{Z}} \sum_{i\geq 0} \binom{n}{i} (-1)^i z_0^{-n} z_1^{n-i} z_2^i. \tag{2.10}$$

We have the following fundamental properties of delta function ([FLM], [FHL], [Le], [Zhu]):

Lemma 2.1 *For $\alpha \in \mathbf{C}$,*

$$z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{\alpha}\delta\left(\frac{z_1-z_2}{z_0}\right)=z_1^{-1}\left(\frac{z_0+z_2}{z_1}\right)^{-\alpha}\delta\left(\frac{z_0+z_2}{z_2}\right); \tag{2.11}$$

For $r, s, k \in \mathbf{Z}$ and for $p(z_1, z_2) \in \mathbf{C}[[\mathbf{z_1}, \mathbf{z_2}]]$,

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)z_1^r z_2^s (z_1-z_2)^k p(z_1,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_0}{-z_0}\right)z_1^r z_2^s(-z_2+z_1)^k p(z_1,z_2) \\ = \; & z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(z_2+z_0)^r z_2^s z_0^k p(z_2+z_0,z_2). \end{aligned} \tag{2.12}$$

In particular,

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)-z_0^{-1}\delta\left(\frac{z_2-z_0}{-z_0}\right)=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right). \tag{2.13}$$

Note that (2.11) is equivalent to

$$(z_1-z_2)^{\alpha}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)=z_1^{-1}z_0^{\alpha}\left(\frac{z_0+z_2}{z_1}\right)^{-\alpha}\delta\left(\frac{z_0+z_2}{z_2}\right) \quad \text{for } \alpha \in \mathbf{C}. \tag{2.14}$$

A generalized vertex algebra by definition is associated to an abelian group G, a symmetric $\mathbf{C}/2\mathbf{Z}$-valued $\mathbf{Z}$-bilinear form (not necessarily nondegenerate) on G:

$$(g,h) \in \mathbf{C}/2\mathbf{Z} \quad \textbf{for } \mathbf{g}, \mathbf{h} \in \mathbf{G} \tag{2.15}$$

and $c(\cdot,\cdot)$ is a $\mathbf{C}^{\times}$-valued alternating $\mathbf{Z}$-bilinear form on G.

A generalized vertex algebra associated with the group G and the forms $(\cdot,\cdot)$ and $c(\cdot,\cdot)$ is a G-graded vector space

$$V=\coprod_{g\in G}V^g, \tag{2.16}$$

equipped with a linear map

$$\begin{aligned} Y: \quad & V \to (\mathrm{End}\, V)\{z\} \\ & u \mapsto Y(u,z)=\sum_{n\in\mathbf{C}}u_n z^{-n-1} \end{aligned} \tag{2.17}$$

and with a distinguished vector $\mathbf{1} \in V^0$, called the *vacuum vector*, satisfying the following conditions for $g, h \in G,\ u, v \in V$ and $l \in \mathbf{C}$:

$$u_l V^h \subset V^{g+h} \quad \text{if } u \in V^g; \tag{2.18}$$

$$u_l v = 0 \quad \text{if the real part of } l \text{ is sufficiently large;} \tag{2.19}$$

$$Y(\mathbf{1},z)=1; \tag{2.20}$$

$$Y(v,z)\mathbf{1} \in V[[z]] \quad \text{and } v_{-1}\mathbf{1} \ (= \lim_{z\to 0} Y(v,z)\mathbf{1}) = v; \tag{2.21}$$

$$Y(u,z)|_{V^h} = \sum_{n\equiv(g,h) \bmod \mathbf{Z}} v_n z^{-n-1} \quad \text{if } u \in V^g \tag{2.22}$$

(i.e., $n+2\mathbf{Z}\equiv(g,h) \bmod \mathbf{Z}/2\mathbf{Z}$);

$$\begin{aligned}
& z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) \\
& -c(g,h)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) \\
= \;& z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)\left(\frac{z_1-z_0}{z_2}\right)^{-g} \qquad (2.23)
\end{aligned}$$

(the *generalized Jacobi identity*) if $u\in V^g,\ v\in V^h$, where

$$\delta\left(\frac{z_1-z_0}{z_2}\right)\left(\frac{z_1-z_0}{z_2}\right)^{-g}\cdot w=\left(\frac{z_1-z_0}{z_2}\right)^{-(g,g')}\delta\left(\frac{z_1-z_0}{z_2}\right)w \qquad (2.24)$$

for $w\in V^{g'},\ g'\in G$.

This completes the definition. The map Y is called the *vertex operator map*. The generalized vertex algebra is denoted by

$$(V,Y,\mathbf{1},G,c(\cdot,\cdot),(\cdot,\cdot))$$

or briefly, by V.

Remark 2.2 Note that we here slightly generalize the original definition in [DL2] where $(\cdot,\cdot)$ was assumed to be $(\frac{1}{T}\mathbf{Z})/\mathbf{2Z}$-valued, where T is a positive integer called the level. The main reason for this generalization is to include the generalized vertex algebras associated to affine algebras with a non-rational level. On the other hand, if G is finite, there exists a positive integer T such that $(\cdot,\cdot)$ ranges in $\frac{1}{T}\mathbf{Z}/2\mathbf{Z}$.

We recall the following remarks from [DL2]:

Remark 2.3 If $G=0$, the notion of generalized vertex algebra reduces to the notion of vertex algebra. If $G=\mathbf{Z}/\mathbf{2Z}$ with $(m+\mathbf{Z},\mathbf{n}+\mathbf{Z})=\mathbf{mn}+\mathbf{Z}$, the notion of generalized vertex algebra reduces to the notion of vertex superalgebra, noting that $c(\cdot,\cdot)=1$.

Remark 2.4 A *generalized vertex operator algebra* is a generalized vertex algebra V associated to a finite group G with $c(\cdot,\cdot)=1$ and $(\cdot,\cdot)$ being nondegenerate and furthermore, it is equipped with another distinguished vector $\omega\in V_2^0$, called the Virasoro vector, such that

$$[L(m),L(n)]=(m-n)L(m+n)+\frac{1}{12}(m^3-m)\delta_{m+n,0}c \quad \text{for } m,n\in\mathbf{Z}, \qquad (2.25)$$

$$[L(-1),Y(v,z)]=\frac{d}{dz}Y(v,z) \quad \text{for } v\in V, \qquad (2.26)$$

where $Y(\omega, z) = \sum_{m\in\mathbf{Z}} L(m)z^{-m-2}$ and c is a complex number, called the *rank* of V, and such that

$$V = \coprod_{n\in\mathbf{C}} V_n \quad \text{where } V_n = \{v \in V \mid L(0)v = nv\} \quad \text{for } n \in \mathbf{C}; \tag{2.27}$$

$$V^g = \coprod_{n\in\mathbf{C}} V_n^g \quad (\text{where } V_n^g = V_n \cap V^g) \quad \text{for } g \in G; \tag{2.28}$$

$$\dim V_n < \infty \quad \text{for } n \in \mathbf{C}, \tag{2.29}$$

$$V_n = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \tag{2.30}$$

Proposition 2.5 *[DL2] In the presence of all the axioms except the generalized Jacobi identity in defining the notion of generalized vertex algebra, the generalized Jacobi identity is equivalent to the following generalized weak commutativity and associativity:*

(A) For $g_1, g_2 \in G$ and $v_1 \in V^{g_1}$, $v_2 \in V^{g_2}$, there exists a nonnegative integer k such that

$$\begin{aligned} &(z_1 - z_2)^{k+(g_1,g_2)} Y(v_1, z_1)Y(v_2, z_2) \\ = \;& (-1)^k c(g_1, g_2)(z_2 - z_1)^{k+(g_1,g_2)} Y(v_2, z_2)Y(v_1, z_1). \end{aligned} \tag{2.31}$$

(B) For $g_1, g_2, h \in G$ and $v_1 \in V^{g_1}$, $v_2 \in V^{g_2}$, $w \in V^h$, there exists a nonnegative integer l such that

$$(z_0 + z_2)^{l+(g_1,h)} Y(v_1, z_0 + z_2)Y(v_2, z_2)w = (z_2 + z_0)^{l+(g_1,h)} Y(Y(v_1, z_0)v_2, z_2)w, \tag{2.32}$$

where l is independent of v_2.

The following result was also due to [DL2]:

Proposition 2.6 *In the presence of all the axioms except the generalized Jacobi identity in defining the notion of generalized vertex algebra, the generalized Jacobi identity follows from the generalized weak commutativity and the following property*

$$[D, Y(v, z)] = \frac{d}{dz} Y(v, z) \quad \textit{for } v \in V, \tag{2.33}$$

where D is an endomorphism of V defined by

$$D(v) = v_{-2}\mathbf{1} \quad \textit{for } v \in V. \tag{2.34}$$

Proof. We here give a slightly different proof by generalizing the proof given in [Li2] for the corresponding result for vertex superalgebras.

First, just as in [Li2], from the vacuum property and (2.33) we have

$$Y(v, z)\mathbf{1} = e^{zD} v \quad \text{for } v \in V. \tag{2.35}$$

Second, (2.33) is equivalent to the following conjugation formula:

$$e^{z_0 D} Y(v,z) e^{-z_0 D} = Y(v, z+z_0). \tag{2.36}$$

Third, we shall derive a skew-symmetry. Let $u \in V^g,\ v \in V^h,\ g,h \in G$. Then there exists a nonnegative integer k such that

$$(z_1 - z_2)^{k+(g,h)} Y(u,z_1)Y(v,z_2) = (-1)^k c(g,h)(z_2 - z_1)^{k+(g,h)} Y(v,z_2)Y(u,z_1) \tag{2.37}$$

and such that

$$z^{k+(g,h)} Y(v,z)u \in V[[z]]. \tag{2.38}$$

Then

$$\begin{aligned} & (z_1 - z_2)^{k+(g,h)} Y(u,z_1)Y(v,z_2)\mathbf{1} \\ = \;& (-1)^k c(g,h)(z_2 - z_1)^{k+(g,h)} Y(v,z_2)Y(u,z_1)\mathbf{1} \\ = \;& (-1)^k c(g,h)(z_2 - z_1)^{k+(g,h)} Y(v,z_2) e^{z_1 D} u \\ = \;& (-1)^k c(g,h) e^{z_1 D} (z_2 - z_1)^{k+(g,h)} Y(v, z_2 - z_1) u \\ = \;& (-1)^k c(g,h) e^{z_1 D} (e^{\pi i} z_1 + z_2)^{k+(g,h)} Y(v, e^{\pi i} z_1 + z_2) u. \end{aligned} \tag{2.39}$$

We are using (2.38). Now, it is safe for us to replace z_2 with 0. In this way we get

$$z_1^{k+(g,h)} Y(u,z_1)v = c(g,h) e^{\pi i (g,h)} z_1^{k+(g,h)} e^{z_1 D} Y(v, e^{\pi i} z_1) u. \tag{2.40}$$

Then we obtain the following skew-symmetry:

$$Y(u,z_1)v = c(g,h) e^{\pi i (g,h)} e^{z_1 D} Y(v, e^{\pi i} z_1) u. \tag{2.41}$$

Next, we prove the generalized weak associativity. Let $u \in V^{g_1},\ v \in V^{g_2},\ w \in V^{g_3}$. Let k be a nonnegative integer (only depending on u, w) such that

$$(z_1 - z_2)^{k+(g_1,g_3)} Y(u,z_1)Y(w,z_2) = (-1)^k c(g_1,g_3)(z_2 - z_1)^{k+(g_1,g_3)} Y(w,z_2)Y(u,z_1). \tag{2.42}$$

Then using the skew-symmetry (2.41) and the conjugation formula (2.36) we obtain the following generalized associativity relation

$$\begin{aligned} & (z_0 + z_2)^{k+(g_1,g_3)} Y(u, z_0 + z_2) Y(v, z_2) w \\ = \;& c(g_2,g_3) e^{\pi i (g_2,g_3)} (z_0 + z_2)^{k+(g_1,g_3)} Y(u, z_0 + z_2) e^{z_2 D} Y(w, e^{\pi i} z_2) v \\ = \;& c(g_2,g_3) e^{\pi i (g_2,g_3)} (z_0 + z_2)^{k+(g_1,g_3)} e^{z_2 D} Y(u, z_0) Y(w, e^{\pi i} z_2) v \\ = \;& (-1)^k c(g_2,g_3) c(g_1,g_3) e^{\pi i (g_2,g_3)} (e^{\pi i} z_2 - z_0)^{k+(g_1,g_3)} e^{z_2 D} Y(w, e^{\pi i} z_2) Y(u, z_0) v \\ = \;& (-1)^k e^{\pi i (g_2,g_3)} (e^{\pi i} z_2 - z_0)^{k+(g_1,g_3)} c(g_1 + g_2, g_3) e^{z_2 D} Y(w, e^{\pi i} z_2) Y(u, z_0) v \\ = \;& (z_2 + z_0)^{k+(g_1,g_3)} Y(Y(u,z_0)v, z_2) w. \end{aligned} \tag{2.43}$$

Then it follows from Proposition 2.5. $\square$

A V-*module* [DL2] is a vector space $W = \coprod_{s\in S} W^s$, where S is a G-set equipped with a $\mathbf{C}/2\mathbf{Z}$-valued function $(\cdot,\cdot)$ on $G \times S$ such that

$$(g_1 + g_2, g_3 + s) = (g_1, g_3) + (g_2, g_3) + (g_1, s) + (g_2, s) \tag{2.44}$$

for $g_1, g_2, g_3 \in G,\ s \in S$, equipped with a vertex operator map Y from V to $(\mathrm{End}\, W)\{z\}$ such that the axioms (2.19), (2.20) and (2.23) hold with suitable changes. Sometimes, to distinguish the vertex operator map Y for a module W from that for the adjoint module V we use notation Y_W.

Using the proof of Lemma 2.2 of [DLM] with a slight modification we get:

Proposition 2.7 *Let W be a V-module. Then on W,*

$$Y(D(v), z) = \frac{d}{dz} Y(v, z) \quad \textit{for } v \in V. \quad \square \tag{2.45}$$

At the end of this section we present the following simple generalization of Lemma 2.3.5 of [Li2]:

Lemma 2.8 *Let $(V, Y, \mathbf{1}, G, c(\cdot,\cdot), (\cdot,\cdot))$ be a generalized vertex algebra and let W be a V-module. Let $u \in V^g,\ v \in V^h,\ n \in \mathbf{Z},\ \mathbf{u_{(i)}} \in \mathbf{V}$ for $i = 1, \ldots, k$. If*

$$\begin{aligned} &(z_1 - z_2)^{n+(g,h)} Y_V(u, z_1) Y_V(v, z_2) - c(g, h)(-1)^n (z_2 - z_1)^{n+(g,h)} Y_V(v, z_2) Y_V(u, z_1) \\ = &\sum_{i=0}^{k} Y_V(u_{(i)}, z_2) \left(\frac{\partial}{\partial z_2}\right)^i \left(z_1^{-1} \delta(z_2/z_1)(z_2/z_1)^g\right) \end{aligned} \tag{2.46}$$

then

$$\begin{aligned} &(z_1 - z_2)^{n+(g,h)} Y_W(u, z_1) Y_W(v, z_2) - c(g, h)(-1)^n (z_2 - z_1)^{n+(g,h)} Y_W(v, z_2) Y_W(u, z_1) \\ = &\sum_{i=0}^{k} Y_W(u_{(i)}, z_2) \left(\frac{\partial}{\partial z_2}\right)^i \left(z_1^{-1} \delta(z_2/z_1)(z_2/z_1)^g\right). \end{aligned} \tag{2.47}$$

Furthermore, if W is a faithful module, the converse is also true. $\square$

Remark 2.9 With Lemma 2.8, we have the following loose statement: If a generalized vertex algebra V is a module for a certain "algebra" with defining relations of type (2.46), then any V-module is also a module for this "algebra." Conversely, if W is a faithful V-module and it is a module for a certain "algebra," then V is also a module for the "algebra." Examples for such "algebras" are affine Lie algebras, the Virasoro algebra, affine Griess algebra [FLM] and Z-algebras.

3 Generalized vertex algebras generated by parafermion operators

Throughout this paper, G is an abelian group, $(\cdot,\cdot)$ is a symmetric $\mathbf{Z}$-bilinear $\mathbf{C}/2\mathbf{Z}$-valued form on G, $c(\cdot,\cdot)$ is a $\mathbf{C}^{\times}$-valued alternating form on G, and S is a G-set equipped with a $\mathbf{C}/2\mathbf{Z}$-valued function denoted also by $(\cdot,\cdot)$ on $G\times S$ which satisfies (2.44).

We fix a choice of representatives (g,h) and (g,s) in $\mathbf{C}$ for $g,h\in G$ and $s\in S$. However, it should be observed that the main notions and results do not depend on this choice.

Let $W=\coprod_{g\in G}W^g$ be a G-graded vector space (over $\mathbf{C}$). By definition,

$$(\mathrm{End}\,W)\{z\}=\left\{a(z)=\sum_{n\in\mathbf{C}}a_nz^{-n-1}\mid a_n\in\mathrm{End}\,W\right\}. \tag{3.1}$$

Following [DL2], for $g\in G$ we define an operator z^g from W to $W\{z\}$ by

$$z^g\cdot w=z^{(g,s)}w\quad\text{for } w\in W^s,\ s\in S. \tag{3.2}$$

Note: This operator of course depends on the choice of representatives of (g,s).

A formal series $a(z)\in(\mathrm{End}\,W)\{z\}$ is said to satisfy *lower truncation condition* if for every $w\in W$, there exist finitely many complex numbers $\alpha_1,\dots,\alpha_r$ such that

$$a(z)w\in z^{\alpha_1}W[[z]]+\cdots+z^{\alpha_r}W[[z]].$$

Clearly, all such series form a subspace of $(\mathrm{End}\,W)\{z\}$.

For $g\in G$, we define $F(W)^g$ to be the vector subspace of $(\mathrm{End}\,W)\{z\}$ consisting of $a(z)$ satisfying the lower truncation condition and

$$z^{(g,s)}a(z)W^s\subset W^{g+s}[[z,z^{-1}]]\quad\text{for } s\in S. \tag{3.3}$$

Note that the notion of $F(W)^g$ does not depend on the choice of representatives (g,s).

Set

$$F(W)=\coprod_{g\in G}F(W)^g. \tag{3.4}$$

(It is a direct sum because W is G-graded.) As indicated in the notions of $F(W)^g$ and $F(W)$, z is treated as a dummy variable, i.e., $a(z),a(z_1)$ and $a(z_2)$ are considered as the same object.

Definition 3.1 Formal series $a(z)\in F(W)^g,\ b(z)\in F(W)^h$ are said to mutually satisfy the *generalized weak commutativity* if there exists a nonnegative integer k such that

$$(z_1-z_2)^{k+(g,h)}a(z_1)b(z_2)=(-1)^kc(g,h)(z_2-z_1)^{k+(g,h)}b(z_2)a(z_1). \tag{3.5}$$

Clearly, this definition is free of the choice of a representative of (g, h). Note that (3.5) also holds if we replace k by any integer greater than k.

Recall that D is the linear endomorphism of $(\mathrm{End}\, W)\{z\}$ such that $D(a(z)) = a'(z)$ for $a(z) \in (\mathrm{End}\, W)\{z\}$, where $a'(z)$ is the formal derivative of $a(z)$. Clearly, D preserves $F(W)$ and its G-grading.

Remark 3.2 It is easy to see that if $a(z)$ and $b(z)$ mutually satisfy the generalized weak commutativity, so do $a'(z)$ and $b(z)$.

A *homogeneous parafermion field operator* on W is a series $a(z)$ of $F(W)^g$ for some $g \in G$ that satisfies the generalized weak commutativity with itself.

Definition 3.3 Let $a(z) \in F(W)^g,\ b(z) \in F(W)^h,\ g, h \in G$. Suppose that $a(z)$ and $b(z)$ satisfy the generalized weak commutativity. Then for each $n \in \mathbf{C}$, we define an element $a(z)_n b(z)$ of $(\mathrm{End}\, W)\{z\}$ by

$$(a(z)_n b(z))w = \mathrm{Res}_{z_0} \mathrm{Res}_{z_1} z_0^n \left(\frac{z + z_0}{z_1}\right)^{-(g,s)} X \tag{3.6}$$

for $w \in W^s,\ s \in S$, where

$$\begin{aligned} X \ = \ & z_0^{-1} \left(\frac{z_1 - z}{z_0}\right)^{(g,h)} \delta\left(\frac{z_1 - z}{z_0}\right) a(z_1) b(z) w \\ & - c(g, h) z_0^{-1} \left(\frac{z - z_1}{z_0}\right)^{(g,h)} \delta\left(\frac{z - z_1}{-z_0}\right) b(z) a(z_1) w. \end{aligned} \tag{3.7}$$

Remark 3.4 Note that because of the generalized weak commutativity, $z_0^{(g,h)} X$, which involves only integral powers of z_0, contains only finitely many negative integral powers of z_0. Then $a(z)_n b(z)$ exists as a formal series for any choice of representatives $(g, h), (g, s)$. Furthermore, since

$$\left(\frac{z_1 - z}{z_0}\right)^m \delta\left(\frac{z_1 - z}{z_0}\right) = \delta\left(\frac{z_1 - z}{z_0}\right), \tag{3.8}$$

$$\left(\frac{z - z_1}{z_0}\right)^{2m} \delta\left(\frac{z - z_1}{-z_0}\right) = \delta\left(\frac{z - z_1}{-z_0}\right) \tag{3.9}$$

for $m \in \mathbf{Z}$, the expression X does not depend on the choice of a representative of (g, h). For $m \geq 0$, we also have

$$\begin{aligned} \left(\frac{z + z_0}{z_1}\right)^m z_0^{-1} \delta\left(\frac{z_1 - z}{z_0}\right) &= z_0^{-1} \delta\left(\frac{z_1 - z}{z_0}\right), \\ \left(\frac{z + z_0}{z_1}\right)^m z_0^{-1} \delta\left(\frac{z - z_1}{-z_0}\right) &= z_0^{-1} \delta\left(\frac{z - z_1}{-z_0}\right). \end{aligned}$$

Then $(a(z)_n b(z))w$ does not depend on the choice of representatives of (g, h) and (g, s).

If $n \notin (g,h)+\mathbf{Z}$, the right-hand side of (3.6) does not involve integral powers of z_0. Therefore,

$$a(z)_n b(z) = 0 \quad \text{for } n \notin (g,h)+\mathbf{Z}. \tag{3.10}$$

For $n \in (g,h)+\mathbf{Z}$, we have

$$\begin{aligned} & (a(z)_n b(z))w \\ = \; & \sum_{i\geq 0} \mathrm{Res}_{z_1} \binom{-(g,s)}{i} z_1^{(g,s)} z^{-(g,s)-i} \cdot \\ & \cdot \left((z_1 - z)^{n+i} a(z_1)b(z)w - (-1)^{n-(g,h)+i} c(g,h)(z-z_1)^{n+i} b(z)a(z_1)w \right). \end{aligned} \tag{3.11}$$

Again, because of the generalized weak commutativity, the sum is really a finite sum. Noticing that $a(z_1)w \in W^{g+s}\{z_1\}$ from (3.11) we have

$$z^{(g,s)+(h,s)}(a(z)_n b(z))w \in W^{g+h+s}((z)).$$

Since $(g+h,s)-(g,s)-(h,s) \in 2\mathbf{Z}$, we have

$$z^{(g+h,s)}(a(z)_n b(z))w \in W^{g+h+s}((z)). \tag{3.12}$$

This shows that $a(z)_n b(z) \in F(W)^{g+h}$.

To summarize we have:

Lemma 3.5 *Suppose that $a(z) \in F(W)^g$, $b(z) \in F(W)^h$ satisfy the generalized weak commutativity. Then*

$$a(z)_n b(z) \in F(W)^{g+h} \quad \textit{for } n \in \mathbf{C}. \tag{3.13}$$

Furthermore, $a(z)_n b(z) = 0$ if $n \notin (g,h)+\mathbf{Z}$, and $a(z)_n b(z) = 0$ for $n \in (g,h)+\mathbf{Z}$ with $n-(g,h)$ being sufficiently large. $\square$

Remark 3.6 Note that in the ordinary untwisted case with $G = 0$, or $\mathbf{Z}/2\mathbf{Z}$ (cf. [Li2]), $a(z)_n b(z)$ were well defined for all $a(z), b(z) \in (\mathrm{End}\, W)[[z, z^{-1}]]$. In the generalized case, the definition of $a(z)_n b(z)$ requires the generalized weak commutativity. This is similar to the situation for twisted vertex operators in [Li3].

Write $a(z)_n b(z)$ in terms of generating series as

$$Y(a(z), z_0)b(z) = \sum_{n\in\mathbf{C}} (a(z)_n b(z))\, z_0^{-n-1}, \tag{3.14}$$

where z_0 is another formal variable. Then

$$Y(a(z),z_0)b(z)=\mathrm{Res}_{z_1}\left(\frac{z+z_0}{z_1}\right)^{-(g,s)}\cdot X. \tag{3.15}$$

Note that $z_0^{(g,h)}Y(a(z),z_0)b(z)$ involves only integral powers of z_0 and that the powers of z_0 are truncated from below. Thus, for $\alpha\in\mathbf{C}$,

$$(z+z_0)^{\alpha}Y(a(z),z_0)b(z)\quad\text{exists}$$

in $F(W)\{z_0,z\}$, hence

$$\left(\frac{z_1-z_0}{z}\right)^{\alpha}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)Y(a(z),z_0)b(z)\quad\text{exists}$$

in $F(W)\{z_0,z_1,z\}$.

As expected we have:

Proposition 3.7 *Let $a(z)\in F(W)^g,\ b(z)\in F(W)^h,\ g,h\in G$. Suppose that $a(z),b(z)$ mutually satisfy the generalized weak commutativity. Then for $w\in W^s,\ s\in S$,*

$$\begin{aligned}&\left(\frac{z_1-z_0}{z}\right)^{-(g,s)}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)(Y(a(z),z_0)b(z))w\\
=\ &z_0^{-1}\left(\frac{z_1-z}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b(z)w\\
&-c(g,h)z_0^{-1}\left(\frac{z-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a(z_1)w.\end{aligned}\tag{3.16}$$

Proof. Let r be a nonnegative integer such that

$$z_1^{r+(g,s)}a(z_1)w\in W[[z_1]],$$

hence

$$\mathrm{Res}_{z_1}z_1^{r+(g,s)}z_0^{-1}\left(\frac{z-z_1}{z_0}\right)^{(g,s)}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a(z_1)w=0. \tag{3.17}$$

Then using (3.15) and the fundamental properties of the delta function, we have

$$\begin{aligned}&(z+z_0)^{r+(g,s)}(Y(a(z),z_0)b(z))w\\
=\ &\mathrm{Res}_{z_1}(z+z_0)^r z_1^{(g,s)}\cdot X\\
=\ &\mathrm{Res}_{z_1}z_1^{r+(g,s)}z_0^{-1}\left(\frac{z_1-z}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b(z)w\\
&-c(g,h)\mathrm{Res}_{z_1}z_1^{r+(g,s)}z_0^{-1}\left(\frac{z-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a(z_1)w\\
=\ &\mathrm{Res}_{z_1}z_1^{r+(g,s)}z_1^{-1}\left(\frac{z_0+z}{z_1}\right)^{-(g,h)}\delta\left(\frac{z_0+z}{z_1}\right)a(z_1)b(z)w\\
=\ &\mathrm{Res}_{z_1}z_1^{r+(g,s)-(g,h+s)+(g,h)}(z_0+z)^{-(g,h)}z_1^{-1}\delta\left(\frac{z_0+z}{z_1}\right)\left(z_1^{(g,h+s)}a(z_1)b(z)w\right)\\
=\ &(z_0+z)^{r+(g,s)}a(z_0+z)b(z)w,\end{aligned}\tag{3.18}$$

noting that $(g,s)-(g,h+s)+(g,h)\in 2\mathbf{Z}$. Similar to the proof of Proposition 2.5, this generalized weak associativity relation together with the generalized weak commutativity relation implies the generalized Jacobi identity. $\square$

Proposition 3.8 *Let $a(z)\in F(W)^{g_1},\ b(z)\in F(W)^{g_2},\ c(z)\in F(W)^{g_3}$ with $g_1,g_2,g_3\in G$. Suppose that $a(z),b(z),c(z)$ mutually satisfy the generalized weak commutativity. Then for $n\in\mathbf{C}$, $a(z)_n b(z)$ and $c(z)$ satisfy the generalized weak commutativity.*

Proof. Let r be a positive integer such that the following identities hold:

$$\begin{aligned}(z_1-z_2)^{r+(g_1,g_3)}a(z_1)c(z_2)&=(-1)^r c(g_1,g_3)(z_2-z_1)^{r+(g_1,g_3)}c(z_2)a(z_1),\\ (z_1-z_2)^{r+(g_2,g_3)}b(z_1)c(z_2)&=(-1)^r c(g_2,g_3)(z_2-z_1)^{r+(g_2,g_3)}c(z_2)b(z_1).\end{aligned}$$

Let $w\in W^s,\ s\in S$. Using the Jacobi identity relation (3.16) and the above generalized weak commutativity relations we get

$$\begin{aligned}
&(z_1-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}\cdot\\
&\cdot\left(\frac{z_1-z_0}{z}\right)^{-(g_1,g_3+s)}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)(Y(a(z),z_0)b(z))c(z_2)w\\
=\ &\cdot z_0^{-1}\left(\frac{z_1-z}{z_0}\right)^{(g_1,g_2)}\delta\left(\frac{z_1-z}{z_0}\right)\cdot\\
&\cdot(z_1-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}a(z_1)b(z)c(z_2)w\\
&-c(g_1,g_2)z_0^{-1}\left(\frac{z-z_1}{z_0}\right)^{(g_1,g_2)}\delta\left(\frac{z-z_1}{-z_0}\right)\cdot\\
&\cdot(z_1-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}b(z)a(z_1)c(z_2)w\\
=\ &c(g_1,g_3)c(g_2,g_3)(z_2-z_1)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\cdot\\
&\cdot z_0^{-1}\left(\frac{z_1-z}{z_0}\right)^{(g_1,g_2)}\delta\left(\frac{z_1-z}{z_0}\right)c(z_2)a(z_1)b(z)w\\
&-c(g_1,g_3)c(g_2,g_3)(z_2-z_1)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\cdot\\
&\cdot c(g_1,g_2)z_0^{-1}\left(\frac{z-z_1}{z_0}\right)^{(g_1,g_2)}\delta\left(\frac{z-z_1}{-z_0}\right)c(z_2)b(z)a(z_1)w\\
=\ &c(g_1,g_3)c(g_2,g_3)(z_2-z_1)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\cdot\\
&\cdot\left(\frac{z_1-z_0}{z}\right)^{-(g_1,s)}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)c(z_2)(Y(a(z),z_0)b(z))w\\
=\ &c(g_1+g_2,g_3)(z_2-z_1)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\cdot\\
&\cdot\left(\frac{z_1-z_0}{z}\right)^{-(g_1,s)}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)c(z_2)(Y(a(z),z_0)b(z))w.
\end{aligned}\tag{3.19}$$

Thus

$$(z_1-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}\cdot$$

$$
\begin{aligned}
&\cdot\left(\frac{z_1-z_0}{z}\right)^{-(g_1,g_3)} z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& c(g_1+g_2,g_3)(z_2-z_1)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\,\cdot \\
&\cdot z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)c(z_2)(Y(a(z),z_0)b(z))w, \qquad (3.20)
\end{aligned}
$$

noting that $(g_1,g_3+s)-(g_1,g_3)-(g_1,s)\in 2\mathbf{Z}$. Using the fundamental properties of delta-function we get

$$
\begin{aligned}
&(z+z_0-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}\,\cdot \\
&\cdot z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& c(g_1+g_2,g_3)(z_2-z-z_0)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}\,\cdot \\
&\cdot z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)c(z_2)(Y(a(z),z_0)b(z))w. \qquad (3.21)
\end{aligned}
$$

Taking Res_{z_1} from (3.21) we obtain

$$
\begin{aligned}
&(z+z_0-z_2)^{r+(g_1,g_3)}(z-z_2)^{r+(g_2,g_3)}(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& c(g_1+g_2,g_3)(z_2-z-z_0)^{r+(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}c(z_2)(Y(a(z),z_0)b(z))w. \quad (3.22)
\end{aligned}
$$

Let $n\in\mathbf{C}$ be arbitrarily fixed. Since $a(z)_n b(z)=0$ for $n\notin(g_1,g_2)+\mathbf{Z}$, we only need to consider $n\in(g_1,g_2)+\mathbf{Z}$. Let N be a fixed nonnegative integer such that $a(z)_m b(z)=0$ for $m\geq N+n$, so that

$$
\mathrm{Res}_{z_0} z_0^{N+n+i}Y(a(z),z_0)b(z)=0 \qquad (3.23)
$$

for $i\in\mathbb{N}$. We may replace r by $r+N$ so that we may assume that $r\geq N$ and that $r+n-(g_1,g_2)>0$. Then using (3.22) we obtain

$$
\begin{aligned}
&(z-z_2)^{3r+(g_1,g_3)+(g_2,g_3)}(a(z)_n b(z))c(z_2)w \\
= \;& \mathrm{Res}_{z_0} z_0^n (z-z_2)^{3r+(g_1,g_3)+(g_2,g_3)}(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& \mathrm{Res}_{z_0}\sum_{i\geq 0}(-1)^i\binom{2r+(g_1,g_3)}{i} z_0^{n+i}(z-z_2+z_0)^{2r+(g_1,g_3)-i}\,\cdot \\
&\cdot(z-z_2)^{r+(g_2,g_3)}(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& \mathrm{Res}_{z_0}\sum_{i=0}^{N}(-1)^i\binom{2r+(g_1,g_3)}{i} z_0^{n+i}\,\cdot \\
&\cdot(z-z_2+z_0)^{2r+(g_1,g_3)-i}(z-z_2)^{r+(g_2,g_3)}(Y(a(z),z_0)b(z))c(z_2)w \\
= \;& \mathrm{Res}_{z_0}\sum_{i=0}^{N}(-1)^i\binom{2r+(g_1,g_3)}{i} z_0^{n+i}c(g_1+g_2,g_3)(-1)^{r-i}\,\cdot \\
&\cdot(z_2-z-z_0)^{2r+(g_1,g_3)-i}(z+z_0)^{(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}c(z_2)(Y(a(z),z_0)b(z))w
\end{aligned}
$$

$$\begin{aligned}
&= \mathrm{Res}_{z_0}\sum_{i\geq 0}(-1)^r\binom{2r+(g_1,g_3)}{i}z_0^{n+i}c(g_1+g_2,g_3)\cdot \\
&\quad \cdot(z_2-z-z_0)^{2r+(g_1,g_3)-i}(z+z_0)^{(g_1,g_3)}(z_2-z)^{r+(g_2,g_3)}c(z_2)(Y(a(z),z_0)b(z))w \\
&= (-1)^r\mathrm{Res}_{z_0}z_0^n c(g_1+g_2,g_3)(z_2-z)^{2r+(g_1,g_3)+(g_2,g_3)}c(z_2)(Y(a(z),z_0)b(z))w \\
&= (-1)^r c(g_1+g_2,g_3)(z_2-z)^{3r+(g_1,g_3)+(g_2,g_3)}c(z_2)(a(z)_n b(z))w. \qquad (3.24)
\end{aligned}$$

Since $(g_1+g_2,g_3)-(g_1,g_3)-(g_2,g_3)\in 2\mathbf{Z}$, there exist $k,k'\in\mathbb{N}$ such that

$$2k+(g_1,g_3)+(g_2,g_3)=(g_1+g_2,g_3)+2k'.$$

Then

$$\begin{aligned}
&(z-z_2)^{3r+2k'+(g_1+g_2,g_3)}(a(z)_n b(z))c(z_2)w \\
= \ &(-1)^r c(g_1+g_2,g_3)(z_2-z)^{3r+2k'+(g_1,+g_2,g_3)}c(z_2)(a(z)_n b(z))w. \qquad (3.25)
\end{aligned}$$

This proves that $a(z)_n b(z)$ and $c(z)$ mutually satisfy the generalized weak commutativity. □

Definition 3.9 A G-graded subspace A of $F(W)$ is called a *generalized vertex pre-algebra* if every pair of homogeneous elements of A satisfy the generalized weak commutativity.

For homogeneous $a(z),b(z)\in A$, $a(z)_n b(z)$ was defined for $n\in\mathbf{C}$. Using linearity, we define $a(z)_n b(z)$ for all $a(z),b(z)\in A$, so that $Y(a(z),z_0)b(z)$ is defined for all $a(z),b(z)\in A$. Furthermore, A is said to be *closed* if $a(z)_n b(z)\in A$ for $a(z),b(z)\in A, n\in\mathbf{C}$. Since the identity operator $I(z)=\mathrm{id}_W$ (independent of z) and any element of $F(W)$ mutually satisfy the generalized weak commutativity, any maximal generalized vertex pre-algebra contains $I(z)$. In view of Proposition 3.8 we immediately have:

Corollary 3.10 *Any maximal generalized vertex pre-algebra V contains the identity operator $I(z)$ and it is closed and D-stable.* □

For the rest of this section, V will be a fixed closed generalized vertex pre-algebra containing the identity operator $I(z)$ on W. The same proof of Lemma 3.10 of [Li3] gives the following results:

Lemma 3.11 *For $a(z)\in V$, we have*

$$Y(I(z),z_0)a(z)=a(z), \qquad (3.26)$$

$$Y(a(z),z_0)I(z)=e^{z_0\frac{\partial}{\partial z}}a(z)=a(z+z_0). \quad □ \qquad (3.27)$$

Furthermore, we have:

Lemma 3.12 *Suppose that $a(z) \in F(W)^g,\ b(z) \in F(W)^h$ mutually satisfy the generalized weak commutativity. Then*

$$\frac{\partial}{\partial z_0} Y(a(z), z_0)b(z) = Y(a'(z), z_0)b(z) = Y(D(a(z)), z_0)b(z), \tag{3.28}$$

$$[D, Y(a(z), z_0)]b(z) = \frac{\partial}{\partial z_0} Y(a(z), z_0)b(z). \tag{3.29}$$

Proof. Let $w \in W^s,\ s \in S$. Recall that the generalized Jacobi relation (3.16) holds. We also have

$$\begin{aligned}
&\left(\frac{z_1 - z_0}{z}\right)^{-(g,s)} z^{-1}\delta\left(\frac{z_1 - z_0}{z}\right)(Y(a'(z), z_0)b(z))w \\
= \ & z_0^{-1}\left(\frac{z_1 - z}{z_0}\right)^{(g,h)} \delta\left(\frac{z_1 - z}{z_0}\right) a'(z_1)b(z)w \\
& - c(g,h) z_0^{-1}\left(\frac{z - z_1}{z_0}\right)^{(g,h)} \delta\left(\frac{z - z_1}{-z_0}\right) b(z)a'(z_1)w.
\end{aligned} \tag{3.30}$$

Let X_L and X_R be the term on the left-hand side and the term on the right-hand side of (3.16). Using (3.30) we get

$$\begin{aligned}
&\left(\frac{z + z_0}{z_1}\right)^{(g,s)} z_1^{-1}\delta\left(\frac{z + z_0}{z_1}\right)(Y(a'(z), z_0)b(z))w \\
= \ & \left(\frac{z_1 - z_0}{z}\right)^{-(g,s)} z^{-1}\delta\left(\frac{z_1 - z_0}{z}\right)(Y(a'(z), z_0)b(z))w \\
= \ & \frac{\partial}{\partial z_1} X_R - a(z_1)b(z)w \frac{\partial}{\partial z_1}\left(z_0^{-1}\left(\frac{z_1 - z}{z_0}\right)^{(g,h)} \delta\left(\frac{z_1 - z}{z_0}\right)\right) \\
& + c(g,h)b(z)a(z_1)w \frac{\partial}{\partial z_1}\left(z_0^{-1}\left(\frac{z - z_1}{z_0}\right)^{(g,h)} \delta\left(\frac{z - z_1}{-z_0}\right)\right) \\
= \ & \frac{\partial}{\partial z_1} X_L + a(z_1)b(z)w \frac{\partial}{\partial z_0}\left(z_0^{-1}\left(\frac{z_1 - z}{z_0}\right)^{(g,h)} \delta\left(\frac{z_1 - z}{z_0}\right)\right) \\
& - c(g,h)b(z)a(z_1)w \frac{\partial}{\partial z_0}\left(z_0^{-1}\left(\frac{z - z_1}{z_0}\right)^{(g,h)} \delta\left(\frac{z - z_1}{-z_0}\right)\right) \\
= \ & \frac{\partial}{\partial z_1} X_L + \frac{\partial}{\partial z_0} X_R \\
= \ & \frac{\partial}{\partial z_1} X_L + \frac{\partial}{\partial z_0} X_L.
\end{aligned} \tag{3.31}$$

We are using the fact

$$\frac{\partial}{\partial z_1}\left(\left(\frac{z - z_1}{z_0}\right)^{\alpha} z_0^{-1}\delta\left(\frac{z - z_1}{z_0}\right)\right) = -\frac{\partial}{\partial z_0}\left(\left(\frac{z - z_1}{z_0}\right)^{\alpha} z_0^{-1}\delta\left(\frac{z - z_1}{z_0}\right)\right). \tag{3.32}$$

for $\alpha \in \mathbf{C}$. Multiplying by $z_1^{(g,s)}$, then taking Res_{z_1} and using a variant of (3.32) we get

$$\begin{aligned}
& (z+z_0)^{(g,s)}(Y(a'(z),z_0)b(z))w \\
= \; & \mathrm{Res}_{z_1} z_1^{(g,s)} \frac{\partial}{\partial z_1} X_L + \mathrm{Res}_{z_1} z_1^{(g,s)} \frac{\partial}{\partial z_0} X_L \\
= \; & \mathrm{Res}_{z_1}(Y(a(z),z_0)b(z))w z_1^{(g,s)} \frac{\partial}{\partial z_1}\left(\left(\frac{z+z_0}{z_1}\right)^{(g,s)} z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)\right) \\
& +\mathrm{Res}_{z_1}(Y(a(z),z_0)b(z))w z_1^{(g,s)} \frac{\partial}{\partial z_0}\left(\left(\frac{z+z_0}{z_1}\right)^{(g,s)} z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)\right) \\
& +\mathrm{Res}_{z_1} z_1^{(g,s)}\left(\frac{z+z_0}{z_1}\right)^{(g,s)} z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)\frac{\partial}{\partial z_0}(Y(a(z),z_0)b(z))w \\
= \; & \mathrm{Res}_{z_1} z_1^{(g,s)}\left(\frac{z+z_0}{z_1}\right)^{(g,s)} z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)\frac{\partial}{\partial z_0}(Y(a(z),z_0)b(z))w \\
= \; & (z+z_0)^{(g,s)}\frac{\partial}{\partial z_0}(Y(a(z),z_0)b(z))w. \qquad (3.33)
\end{aligned}$$

Multiplying by $(z+z_0)^{-(g,s)}$ from left we get the first identity.

The second identity follows from a similar argument and the fact:

$$\frac{\partial}{\partial z_1}\left(\left(\frac{z_1-z}{z_0}\right)^{\alpha} z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right) = -\frac{\partial}{\partial z}\left(\left(\frac{z_1-z}{z_0}\right)^{\alpha} z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right) \qquad (3.34)$$

for $\alpha \in \mathbf{C}$. $\square$

Proposition 3.13 *Let V be a closed generalized vertex pre-algebra of parafermion operators on W. Let $a(z) \in V^{g_1},\ b(z) \in V^{g_2}$ and let $r \in \mathbb{N}$ be such that*

$$(z_1-z_2)^{r+(g_1,g_2)}a(z_1)b(z_2) = (-1)^r c(g_1,g_2)(z_2-z_1)^{r+(g_1,g_2)}b(z_2)a(z_1). \qquad (3.35)$$

Then

$$\begin{aligned}
& (x_1-x_2)^{r+(g_1,g_2)}Y(a(z),x_1)Y(b(z),x_2) \\
= \; & (-1)^r c(g_1,g_2)(x_2-x_1)^{r+(g_1,g_2)}Y(b(z),x_2)Y(a(z),x_1), \qquad (3.36)
\end{aligned}$$

acting on V.

Proof. Let $c(z) \in V^{g_3},\ w \in W^s,\ g_3 \in G,\ s \in S$. Using the generalized Jacobi relation (3.16) and the fundamental properties of the delta-function we get

$$\begin{aligned}
& \left(\frac{z+x_1}{z_1}\right)^{(g_1,s)} z_1^{-1}\delta\left(\frac{z+x_1}{z_1}\right)\left(\frac{z+x_2}{z_2}\right)^{(g_2,s)} z_2^{-1}\delta\left(\frac{z+x_2}{z_2}\right)\cdot \\
& \cdot(Y(a(z),x_1)Y(b(z),x_2)c(z))w
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{z_1-x_1}{z}\right)^{-(g_1,s)} z^{-1}\delta\left(\frac{z_1-x_1}{z}\right)\left(\frac{z_2-x_2}{z}\right)^{-(g_2,s)} z^{-1}\delta\left(\frac{z_2-x_2}{z}\right)\cdot \\
&\quad \cdot(Y(a(z),x_1)Y(b(z),x_2)c(z))w \\
&= \left(\frac{z_2-x_2}{z}\right)^{-(g_2,s)} z^{-1}\delta\left(\frac{z_2-x_2}{z}\right)\cdot \\
&\quad \cdot x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)a(z_1)(Y(b(z),x_2)c(z))w \\
&\quad -c(g_1,g_2+g_3)\left(\frac{z_2-x_2}{z}\right)^{-(g_2,s)} z^{-1}\delta\left(\frac{z_2-x_2}{z}\right)\cdot \\
&\quad \cdot x_1^{-1}\left(\frac{z-z_1}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z-z_1}{-x_1}\right)(Y(b(z),x_2)c(z))a(z_1)w \\
&= x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
&\quad \cdot x_2^{-1}\left(\frac{z_2-z}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z_2-z}{x_2}\right)a(z_1)b(z_2)c(z)w \\
&\quad -x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
&\quad \cdot c(g_2,g_3)x_2^{-1}\left(\frac{z-z_2}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z-z_2}{-x_2}\right)a(z_1)c(z)b(z_2)w \\
&\quad -c(g_1,g_2+g_3)x_1^{-1}\left(\frac{z-z_1}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z-z_1}{-x_1}\right)\cdot \\
&\quad \cdot\left(\frac{z_2-x_2}{z}\right)^{-(g_2,s)} z^{-1}\delta\left(\frac{z_2-x_2}{z}\right)(Y(b(z),x_2)c(z))a(z_1)w. \qquad (3.37)
\end{aligned}
$$

Let $p,q\in\mathbb{N}$ be such that

$$
z_1^{p+(g_1,s)}a(z_1)w\in W[[z_1]],\quad z_2^{q+(g_2,s)}b(z_2)w\in W[[z_2]]. \qquad (3.38)
$$

Notice that for $j\in\mathbb{N}$ we have

$$
(z+x_3)^jA=z_3^jA, \qquad (3.39)
$$

where A is one of the three delta-functions $x_3^{-1}\delta\left(\frac{z_3-z}{x_3}\right)$, $x_3^{-1}\delta\left(\frac{z-z_3}{-x_3}\right)$ and $z^{-1}\delta\left(\frac{z_3-x_3}{z}\right)$. Applying $\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(z+x_1)^s(z+x_2)^qz_1^{(g_1,s)}z_2^{(g_2,s)}$ to (3.37), then using (3.39) and (3.38) we get

$$
\begin{aligned}
&(z+x_1)^{p+(g_1,h)}(z+x_2)^{q+(g_2,s)}(Y(a(z),x_1)Y(b(z),x_2)c(z))w \\
&= \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{p+(g_1,h)}z_2^{q+(g_2,s)}x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
&\quad \cdot x_2^{-1}\left(\frac{z_2-z}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z_2-z}{x_2}\right)a(z_1)b(z_2)c(z)w
\end{aligned}
$$

$$
\begin{aligned}
& -\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{p+(g_1,s)} x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
& \cdot c(g_2,g_3) x_2^{-1}\left(\frac{z-z_2}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z-z_2}{-x_2}\right) a(z_1)c(z)\left(z_2^{q+(g_2,s)}b(z_2)w\right) \\
& -\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_2^{q+(g_2,s)} c(g_1,g_2+g_3) x_1^{-1}\left(\frac{z-z_1}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z-z_1}{-x_1}\right)\cdot \\
& \cdot\left(\frac{z_2-x_2}{z}\right)^{-(g_2,s)} z^{-1}\delta\left(\frac{z_2-x_2}{z}\right)\left(Y(b(z),x_2)c(z)\right)\left(z_1^{p+(g_1,s)}a(z_1)w\right) \\
= \; & \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{p+(g_1,s)} z_2^{q+(g_2,s)} x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_2+g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
& \cdot x_2^{-1}\left(\frac{z_2-z}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z_2-z}{x_2}\right) a(z_1)b(z_2)c(z)w. \qquad (3.40)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& (x_1-x_2)^{r+(g_1,g_2)} x_1^{-1}\delta\left(\frac{z_1-z}{x_1}\right) x_2^{-1}\delta\left(\frac{z_2-z}{x_2}\right) \\
= \; & (x_1-z_2+z)^{r+(g_1,g_2)} x_1^{-1}\delta\left(\frac{z_1-z}{x_1}\right) x_2^{-1}\delta\left(\frac{z_2-z}{x_2}\right) \\
= \; & \left(\frac{z_1-z}{x_1}\right)^{-r-(g_1,g_2)} (z_1-z_2)^{r+(g_1,g_2)} x_1^{-1}\delta\left(\frac{z_1-z}{x_1}\right) x_2^{-1}\delta\left(\frac{z_2-z}{x_2}\right) \\
= \; & \left(\frac{z_1-z}{x_1}\right)^{-(g_1,g_2)} (z_1-z_2)^{r+(g_1,g_2)} x_1^{-1}\delta\left(\frac{z_1-z}{x_1}\right) x_2^{-1}\delta\left(\frac{z_2-z}{x_2}\right). \qquad (3.41)
\end{aligned}
$$

Then

$$
\begin{aligned}
& (z+x_1)^{p+(g_1,s)}(z+x_2)^{q+(g_2,s)}(x_1-x_2)^{r+(g_1,g_2)}(Y(a(z),x_1)Y(b(z),x_2)c(z))w \\
= \; & \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{p+(g_1,s)} z_2^{q+(g_2,s)} x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_3)}\delta\left(\frac{z_1-z}{x_1}\right)\cdot \\
& \cdot x_2^{-1}\left(\frac{z_2-z}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z_2-z}{x_2}\right)(z_1-z_2)^{r+(g_1,g_2)} a(z_1)b(z_2)c(z)w. \qquad (3.42)
\end{aligned}
$$

Using the obvious symmetry, we have

$$
\begin{aligned}
& (z+x_1)^{p+(g_1,s)}(z+x_2)^{q+(g_2,s)}(x_2-x_1)^{r+(g_1,g_2)}(Y(b(z),x_2)Y(a(z),x_1)c(z))w \\
= \; & \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{p+(g_1,s)} z_2^{q+(g_2,s)} x_2^{-1}\left(\frac{z_2-z}{x_2}\right)^{(g_2,g_3)}\delta\left(\frac{z_2-z}{x_2}\right)\cdot \\
& \cdot x_1^{-1}\left(\frac{z_1-z}{x_1}\right)^{(g_1,g_3)}\delta\left(\frac{z_1-z}{x_1}\right)(z_2-z_1)^{r+(g_1,g_2)} b(z_2)a(z_1)c(z)w. \qquad (3.43)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& (z+x_1)^{p+(g_1,s)}(z+x_2)^{q+(g_2,s)}(x_1-x_2)^{r+(g_1,g_2)}(Y(a(z),x_1)Y(b(z),x_2)c(z))w \\
= \; & (-1)^r c(g_1,g_2)(z+x_1)^{p+(g_1,s)}(z+x_2)^{q+(g_2,s)}(x_2-x_1)^{r+(g_1,g_2)}\cdot \\
& \cdot(Y(b(z),x_2)Y(a(z),x_1)c(z))w. \qquad (3.44)
\end{aligned}
$$

Multiplying both sides by $(z+x_1)^{-p-(g_1,s)}(z+x_2)^{-q-(g_2,s)}$ we obtain

$$\begin{aligned}&(x_1-x_2)^{r+(g_1,g_2)}(Y(a(z),x_1)Y(b(z),x_2)c(z))w\\ =\;&(-1)^r c(g_1,g_2)(x_2-x_1)^{r+(g_1,g_2)}(Y(b(z),x_2)Y(a(z),x_1)c(z))w.\end{aligned}\tag{3.45}$$

Then the generalized weak commutativity (3.36) follows immediately. $\square$

Now, we are in a position to present our main theorem:

Theorem 3.14 *Let V be a closed generalized vertex pre-algebra of parafermionic field operators on W, containing the identity operator $I(z)$. Then V is a generalized vertex algebra and W is a canonical V-module with $Y(a(z),z_1)=a(z_1)$ for $a(z)\in V$.*

Proof. It follows from Propositions 2.6, 3.13 and Lemmas 3.5, 3.11 and 3.12 that (V,Y) is a generalized vertex algebra. It follows from Proposition 3.7 that W is a V-module under the natural action. $\square$

The following is a very useful consequence:

Theorem 3.15 *Let Γ be a set of homogeneous parafermion field operators on W that satisfy the generalized weak commutativity. Then the subspace $\langle\Gamma\rangle$ of $F(W)$, linearly spanned by*

$$a^{(1)}(z)_{n_1}\cdots a^{(r)}(z)_{n_r}I(z)\tag{3.46}$$

for $r\in\mathbb{N}$, $a^{(i)}(z)\in\Gamma$, $n_1,\dots,n_r\in\mathbf{C}$, equipped with the vertex operator map Y is a generalized vertex algebra with W as a natural module.

Proof. By Zorn's lemma, there exists a maximal generalized vertex pre-algebra V containing Γ. By Corollary 3.10, V is closed and contains $I(z)$. By Theorem 3.14, V is a generalized vertex algebra. It follows from Proposition 14.8 of [DL2] that $\langle\Gamma\rangle$ is a generalized vertex subalgebra of V. Clearly, $\langle\Gamma\rangle$ does not depend on the choice of V and it is the intersection of all closed generalized vertex pre-algebra containing Γ and the identity operator $I(z)$. $\square$

Lemma 3.16 *Let V be a generalized vertex algebra and let U be a graded subspace which generates V. Let W be a V-module and let $e\in W^0$ such that $Y(u,z)e\in V[[z]]$ for $u\in U$. Then*

$$Y(v,z)e\in W[[z]]\quad\textit{for all } v\in V.\tag{3.47}$$

Proof. Let V' be the collection of all $v\in V$ such that $Y(v,z)e\in V[[z]]$. Clearly, V' is a graded subspace of V containing $\mathbf{1}$ and U. Let $u\in(V')^g$, $v\in(V')^h$, $g,h\in G$. Then

$Y(u,z)e, Y(v,z)e \in V[[z]]$. Applying the generalized Jacobi identity to e, then taking Res_{z_1} we get

$$\begin{aligned}
& Y(Y(u,z_0)v,z_2)e \\
= \ & \mathrm{Res}_{z_1} z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)e \\
& -\mathrm{Res}_{z_1} c(g,h) z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)e \\
= \ & \mathrm{Res}_{z_1} z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)e \\
\in \ & (V((z_0)))[[z_2]]. \qquad (3.48)
\end{aligned}$$

Then

$$Y(u_n v, z_2)e \in V[[z_2]] \quad \text{for } n \in \mathbf{C}. \qquad (3.49)$$

Thus $u_n v \in V'$. Thus, V' is a subalgebra of V, containing $\mathbf{1}$ and U. Consequently, $V' = V$. The proof is complete. $\square$

The same proof of Proposition 3.4 of [Li1] gives:

Proposition 3.17 *Let V be a generalized vertex algebra, let W be a V-module and let $e \in W^0$ such that $Y(v,z)e \in V[[z]]$ for $v \in V$. Then*

$$Y(v,z)e = e^{zL(-1)}v_{-1}e \quad \textit{for } v \in V, \qquad (3.50)$$

and the linear map

$$\begin{aligned}
f: \quad & V \to W \\
& v \mapsto v_{-1}e \qquad (3.51)
\end{aligned}$$

is a V-homomorphism. Furthermore, if W is faithful and generated by e, then f is an isomorphism. $\square$

Now we have the following result the first part of which is a generalization of a result obtained in [FKRW] and [PM]:

Theorem 3.18 *Let V be a G-graded vector space and $\mathbf{1} \in V^0$ and let U a generalized vertex pre-algebra of parafermions on V such that $\psi(z)\mathbf{1} \in V[[z]]$ and such that V is generated from $\mathbf{1}$ by all component operators of $\psi(z)z^g$ for $\psi(z) \in U^g$, $g \in G$. Then there exists a unique generalized vertex algebra structure on V such that $\mathbf{1}$ is the vacuum vector and that $Y(a,z) = a(z)$ for $a \in U$. Furthermore, this generalized vertex algebra V is isomorphic to the generalized vertex algebra generated by U inside* $(\mathrm{End}\, V)\{z\}$.

Proof. Let $\bar{U}$ be the generalized vertex algebra generated by U inside $F(V)$. Then V is a $\bar{U}$-module and $Y(u,z)\mathbf{1} \in V[[z]]$ for $u \in U$. Since V is a faithful $\bar{U}$-module, it follows from Lemma 3.16 and Proposition 3.17 that the linear map f from $\bar{U}$ to V such that $f(u) = u_{-1}\mathbf{1}$ for $u \in \bar{U}$ is an one-to-one $\bar{U}$-homomorphism. Clearly, f is onto. Then $V = \bar{U}$ has a natural generalized vertex algebra structure. The other assertions are clear. $\square$

Similar to Lemma 3.16 we have the following result:

Lemma 3.19 *Let V be a generalized vertex algebra with a graded generating subspace A, W be a V-module and D_W be a grading-preserving endomorphism of W such that*

$$[D_W, Y(v,z)] = \frac{d}{dz}Y(v,z) \; (= Y(D(v),z)) \quad \text{for } v \in A. \tag{3.52}$$

Then (3.52) for all $v \in V$.

Proof. Recall that $Y(D(v),z) = \frac{d}{dz}Y(v,z)$ for all $v \in V$. Set

$$K = \{v \in V \mid [D_W, Y(v,z)] = Y(D(v),z) = \frac{d}{dz}Y(v,z)\}. \tag{3.53}$$

Then $A \subset K$.

Let $u \in K \cap V^g,\ v \in K \cap V^h,\ g,h \in G$ and let $w \in W^s,\ s \in S$. Then we have

$$\begin{aligned}
& z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-(g,s)}\delta\left(\frac{z_1-z_0}{z_2}\right)[D_W, Y(Y(u,z_0)v,z_2)]w\\
= \; & z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)[D_W, Y(u,z_1)]Y(v,z_2)w\\
& +z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)[D_W, Y(v,z_2)]w\\
& -c(g,h)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)[D_W, Y(v,z_2)]Y(u,z_1)w\\
& -c(g,h)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)[D_W, Y(u,z_1)]w\\
= \; & z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(D(u),z_1)Y(v,z_2)w\\
& +z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(D(v),z_2)w\\
& -c(g,h)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(D(v),z_2)Y(u,z_1)w\\
& -c(g,h)z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(D(u),z_1)w
\end{aligned}$$

$$
\begin{aligned}
&= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-(g,s)}\delta\left(\frac{z_1-z_0}{z_2}\right)(Y(Y(D(u),z_0)v,z_2)+Y(Y(u,z_0)D(v),z_2))w\\
&= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-(g,s)}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(DY(u,z_0)v,z_2)w. \qquad (3.54)
\end{aligned}
$$

This gives

$$
[D_W, Y(Y(u,z_0)v,z_2)]w = Y(DY(u,z_0)v,z_2)w. \qquad (3.55)
$$

Then $Y(u,z_0)v \in K\{z_0\}$. Thus K is a subalgebra of V containing A and $\mathbf{1}$. Since A generates V, we must have $K = V$. This concludes the proof. $\square$

With Lemma 3.19 we immediately have:

Proposition 3.20 *Let A be a generalized vertex pre-algebra of parafermions on W and let D_W be a grading-preserving endomorphism of W such that*

$$
[D_W, a(z)] = \frac{d}{dz}a(z) \quad \textit{for } a(z) \in A. \qquad (3.56)
$$

Denote by V the generalized vertex algebra generated by A. Then on W,

$$
[D_W, Y(v,z)] = Y(D(v),z) = \frac{d}{dz}Y(v,z) \quad \textit{for } v \in V. \quad \square \qquad (3.57)
$$

4 Generalized vertex algebras associated with Z-algebras

In this section we briefly recall from [LW1-3] and [LP1-2] the fundamental results about Z-operators and we then show that the vacuum space $\Omega_{M(\ell,0)}$ of the generalized Verma module $M(\ell,0)$ for an affine Lie algebra $\hat{\mathfrak{g}}$ with respect to the homogeneous Heisenberg subalgebra has a canonical generalized vertex algebra structure. We also show that for any highest weight $\hat{\mathfrak{g}}$-module W of level ℓ, Ω_W is an $\Omega_{M(\ell,0)}$-module.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathbf{h}$ be a Cartan subalgebra, Φ be the set of roots and $\mathbb{Q}$ be the root lattice. Let $\langle\cdot,\cdot\rangle$ be the normalized killing form such that $\langle\alpha,\alpha\rangle = 2$ for a long root α. Using $\langle\cdot,\cdot\rangle$ we identify $\mathbf{h}^*$ with $\mathbf{h}$.

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra:

$$\hat{\mathfrak{g}} = \mathfrak{g}\otimes \mathbf{C}[\mathbf{t},\mathbf{t^{-1}}]\oplus \mathbf{Cc}, \tag{4.1}$$

where

$$[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + m\langle a,b\rangle\delta_{m+n,0}c, \tag{4.2}$$

$$[\hat{\mathfrak{g}}, c] = 0 \tag{4.3}$$

for $a,b\in\mathfrak{g},\ m,n\in\mathbf{Z}$. Following the tradition, we also use $a(n)$ for $a\otimes t^n$. For $n\in\mathbf{Z}$, we denote

$$\mathfrak{g}(n) = \{a(n)\mid a\in\mathfrak{g}\}. \tag{4.4}$$

For $a\in\mathfrak{g}$, define the generating series

$$a(z) = \sum_{n\in\mathbf{Z}}(a\otimes t^n)z^{-n-1}\in\hat{\mathfrak{g}}[[z,z^{-1}]]. \tag{4.5}$$

Set

$$\hat{\mathbf{h}}^+ = \mathbf{h}\otimes t\mathbf{C}[\mathbf{t}],\quad \hat{\mathbf{h}}^- = \mathbf{h}\otimes\mathbf{t^{-1}}\mathbf{C}[\mathbf{t^{-1}}], \tag{4.6}$$

subalgebras of $\hat{\mathbf{h}} = \mathbf{h}\otimes\mathbf{C}[\mathbf{t},\mathbf{t^{-1}}]+\mathbf{Cc}$. Then

$$\hat{\mathbf{h}}_{\mathbf{Z}} = \hat{\mathbf{h}}^+ + \hat{\mathbf{h}}^- + \mathbf{Cc} \tag{4.7}$$

is a Heisenberg subalgebra of $\hat{\mathfrak{g}}$. For $\ell\in\mathbf{C}^\times$, let $M(\ell)$ be the standard irreducible $\hat{\mathbf{h}}_{\mathbf{Z}}$-module with c acting as scalar ℓ. We may also consider $M(\ell)$ as an $\hat{\mathbf{h}}$-module with the action of $\mathbf{h}$ being zero.

Definition 4.1 For $\ell\in\mathbf{C}$, we define a category $\mathcal{C}_\ell$ whose objects are level-ℓ $\hat{\mathfrak{g}}$-modules W which are $\mathbf{h}$-weight modules satisfying the condition that for every $w\in W$, $\mathfrak{g}(n)w=0$ for n sufficiently large and $\dim U(\hat{\mathbf{h}}^+)w<\infty$.

It follows from [LW3] and [K] that each W from $\mathcal{C}_\ell$ with $\ell \neq 0$ is a completely reducible $\hat{\mathbf{h}}$-module. For $W \in \mathcal{C}_\ell$, set

$$\Omega_W = \{w \in W \mid h(n)w = 0 \quad \text{for } h \in \mathbf{h},\ n \in \mathbf{Z}_+\}. \tag{4.8}$$

Since $[h(0), h'(m)] = 0$ for $h, h' \in \mathbf{h},\ m \in \mathbf{Z}$, $h(0)$ preserves Ω_W. With $\hat{\mathfrak{g}}$, as an $\mathbf{h}$-module, being naturally $\mathbf{h}$ $(= \mathbf{h}^*)$-graded we have

$$\Omega_W = \coprod_{\alpha \in \mathbf{h}} \Omega_W^\alpha, \tag{4.9}$$

where $\Omega_W^\alpha = \{w \in \Omega_W \mid h(0)w = \langle \alpha, h\rangle w \quad \text{for } h \in \mathbf{h}\}$.

For $h \in \mathbf{h}$, set ([LW1])

$$E^\pm(h, z) = \exp\left(\sum_{\pm n \in \mathbf{Z}_+} \frac{h(n)}{n} z^{-n}\right) \in U(\hat{\mathfrak{g}})[[z^{\mp 1}]]. \tag{4.10}$$

For $a \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$, we define

$$Z(a, z) = E^-(\frac{1}{\ell}\alpha, z) a(z) E^+(\frac{1}{\ell}\alpha, z), \tag{4.11}$$

a formal object. (It is an element of $\overline{U(\hat{\mathfrak{g}})}[[z, z^{-1}]]$, where $\overline{U(\hat{\mathfrak{g}})}$ is a certain formal completion of $U(\hat{\mathfrak{g}})$.) For every $W \in \mathcal{C}_\ell$, $Z(v, z)$ is a well defined element of $(\text{End } W)[[z, z^{-1}]]$. Then we have ([LW4], [LP2]):

Proposition 4.2 *Let $\ell \in \mathbf{C}^\times$, $\mathbf{W} \in \mathcal{C}_\ell$. For $h \in \mathbf{h},\ u \in \mathfrak{g}_\alpha,\ v \in \mathfrak{g}_\beta,\ \alpha, \beta \in \Phi$, on W,*

$$[h(0), Z(u, z)] = \langle \alpha, h\rangle Z(u, z), \tag{4.12}$$

$$[h(n), Z(u, z)] = 0 \quad \textit{for } n \neq 0, \tag{4.13}$$

$$\begin{aligned} &(1 - z_2/z_1)^{\langle \alpha, \beta\rangle/\ell} Z(u, z_1) Z(v, z_2) - (1 - z_1/z_2)^{\langle \alpha, \beta\rangle/\ell} Z(v, z_2) Z(u, z_1) = \\ &= \begin{cases} z_1^{-1}\delta(z_2/z_1) Z([u, v], z_2) & \textit{if } \alpha + \beta \neq 0, \\ z_1^{-1}\delta(z_2/z_1)[u, v] z_2^{-1} + \ell\langle u, v\rangle \frac{\partial}{\partial z_2} z_1^{-1}\delta(z_2/z_1) & \textit{if } \alpha + \beta = 0. \end{cases} \end{aligned} \tag{4.14}$$

It follows immediately from (4.13) that $Z(v, z)$ maps Ω_W to $\Omega_W[[z, z^{-1}]]$.

Set

$$Z(v, z) = \sum_{n \in \mathbf{Z}} Z(v, n) z^{-n}. \tag{4.15}$$

Definition 4.3 Following [LP2] we define a category $\mathcal{Z}_\ell$ (which was denoted by P_ℓ in [LP2]) whose objects are $\mathbf{h}$-weight modules U equipped with a family of operators $Z_U(a, m)$ (linear in a) on U for $a \in \mathfrak{g}_\alpha,\ \alpha \in \Phi,\ m \in \mathbf{Z}$ such that $Z_U(a, z)w \in U((z))$ and such that (4.12) and (4.14) hold for $Z_W(u, z)$ in place of $Z(u, z)$.

Clearly, we have a functor Ω from $\mathcal{C}_\ell$ to $\mathcal{Z}_\ell$. Conversely, given $U \in \mathcal{Z}_\ell$, we set

$$E(U) = M(\ell) \otimes U \quad \text{as a vector space.} \tag{4.16}$$

View $E(U)$ as a natural $\hat{\mathbf{h}}_{\mathbf{Z}}$-module with $h(n)$ (for $n \neq 0$) acting on the first factor. Then define an action of $\hat{\mathfrak{g}}$ by

$$c \mapsto \ell, \tag{4.17}$$

$$h \mapsto 1 \otimes h \quad \text{for } h \in \mathbf{h}, \tag{4.18}$$

$$a(z) \mapsto E^-(-\frac{1}{\ell}\alpha, z)E^+(-\frac{1}{\ell}\alpha, z) \otimes Z_U(a, z) \quad \text{for } a \in \mathfrak{g}_\alpha,\ \alpha \in \Phi. \tag{4.19}$$

It was proved in [LP2] that $E(U)$ is a $\hat{\mathfrak{g}}$-module in $\mathcal{C}_\ell$. Furthermore, we have ([LW4], [LP2]):

Proposition 4.4 *Let $\ell \in \mathbf{C}^\times$. Then the functors*

$$\Omega : W \mapsto \Omega_W \ \ \textit{and} \ \ E : U \mapsto E(U) \tag{4.20}$$

are exact and they define equivalences between the categories $\mathcal{C}_\ell$ and $\mathcal{Z}_\ell$. In particular, W is irreducible in $\mathcal{C}_\ell$ if and only if Ω_W is irreducible in $\mathcal{Z}_\ell$.

For $\lambda \in \mathbf{h}$, let $M(\ell, \lambda)$ be the Verma $\hat{\mathfrak{g}}$-module. In view of the universal property for $M(\ell, \lambda)$, with Proposition 4.4 we immediately have:

Corollary 4.5 *Let $\ell \in \mathbf{C}$, $\lambda \in \mathbf{h}$ and let v be a (nonzero) highest weight vector of $M(\ell, \lambda)$. Let $U \in \mathcal{Z}_\ell$ and let $e \in U$ satisfying the following conditions:*

$$he = \langle h, \lambda\rangle e \quad \textit{for } h \in \mathbf{h}, \tag{4.21}$$

$$Z_U(u, z)e \in z^{-1}U[[z]] \quad \textit{for } u \in \mathfrak{g}_\alpha,\ \alpha \in \Phi, \tag{4.22}$$

$$Z_U(v, z)e \in U[[z]] \quad \textit{for } v \in \mathfrak{g}_\beta,\ \beta \in \Phi_+. \tag{4.23}$$

Then there exists a unique morphism in $\mathcal{Z}_\ell$ from $\Omega_{M(\ell,\lambda)}$ to U sending v to e. □

Definition 4.6 For $a \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$, we define

$$\psi(a, z) = Z(a, z)z^{-\frac{1}{\ell}\alpha(0)} = E^-(\frac{1}{\ell}\alpha, z)a(z)E^+(\frac{1}{\ell}\alpha, z)z^{-\frac{1}{\ell}\alpha(0)}. \tag{4.24}$$

Then

$$[h(n), \psi(a, z)] = 0 \quad \text{for } h \in \mathbf{h},\ n \neq 0, \tag{4.25}$$

hence $\psi(a, z)$ maps Ω_W to $\Omega_W\{z\}$ for $W \in \mathcal{C}_\ell$. Note that (4.12) amounts to

$$z^{h(0)}Z(u, z_1) = Z(u, z_1)z^{\langle\alpha,h\rangle + h(0)} \tag{4.26}$$

for $h \in \mathbf{h},\ u \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$. We have the following reformulation of Proposition 4.2 in terms of ψ-operators:

Proposition 4.7 *Let $u \in \mathfrak{g}_\alpha,\ v \in \mathfrak{g}_\beta,\ \alpha, \beta \in \Phi,\ \ell \in \mathbf{C}^\times$ and $W \in \mathcal{Z}_\ell$. On W,*

$$[h(0), \psi(u,z)] = \langle \alpha, h\rangle \psi(u,z), \tag{4.27}$$

$$[h(n), \psi(u,z)] = 0 \quad \text{for } h \in \mathbf{h},\ n \neq 0, \tag{4.28}$$

$$(z_1 - z_2)^{\langle \alpha,\beta\rangle/\ell} \psi(u,z_1)\psi(v,z_2) - (z_2 - z_1)^{\langle \alpha,\beta\rangle/\ell} \psi(v,z_2)\psi(u,z_1) =$$

$$= \begin{cases} z_1^{-1}\delta(z_2/z_1)\psi([u,v],z_2)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)} & \text{if } \alpha+\beta \neq 0, \\ \ell\langle u,v\rangle \frac{\partial}{\partial z_2}\left(z_1^{-1}\delta(z_2/z_1)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)}\right) & \text{if } \alpha+\beta = 0. \end{cases} \tag{4.29}$$

Proof. The first two identities are obvious. Using (4.26) we obtain

$$(z_1 - z_2)^{\langle \alpha,\beta\rangle/\ell} \psi(u,z_1)\psi(v,z_2) - (z_2 - z_1)^{\langle \alpha,\beta\rangle/\ell} \psi(v,z_2)\psi(u,z_1) =$$

$$= \begin{cases} z_1^{-1}\delta(z_2/z_1)\psi([u,v],z_2)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)} & \text{if } \alpha+\beta \neq 0, \\ \left(z_1^{-1}\delta(z_2/z_1)[u,v]z_2^{-1} + \ell\langle u,v\rangle \frac{\partial}{\partial z_2} z_1^{-1}\delta(z_2/z_1)\right)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)} & \text{if } \alpha+\beta = 0. \end{cases} \tag{4.30}$$

It remains to consider the case $\alpha + \beta = 0$. Using the fact that $[u,v] = \langle u,v\rangle\alpha$ and $\delta(z)z^m = \delta(z)$ for $m \in \mathbf{Z}$ we obtain

$$\begin{aligned} &\left(z_1^{-1}\delta(z_2/z_1)[u,v]z_2^{-1} + \ell\langle u,v\rangle \frac{\partial}{\partial z_2} z_1^{-1}\delta(z_2/z_1)\right)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)} \\ = \;\; &\ell\langle u,v\rangle \frac{\partial}{\partial z_2}\left(z_1^{-1}\delta(z_2/z_1)(z_2/z_1)^{\frac{1}{\ell}\alpha(0)}\right). \end{aligned} \tag{4.31}$$

This completes the proof. $\square$

It is a simple fact (see for example [Li2]) that

$$(z_1 - z_2)^m \left(\frac{\partial}{\partial z_2}\right)^n z_2^{-1}\delta(z_1/z_2) = 0 \tag{4.32}$$

for $m, n \in \mathbf{Z}$ with $m > n \geq 0$. Then using Proposition 4.7 we get

$$(z_1 - z_2)^2 \left((z_1 - z_2)^{\langle \alpha,\beta\rangle/\ell} \psi(u,z_1)\psi(v,z_2) - (z_2 - z_1)^{\langle \alpha,\beta\rangle/\ell} \psi(v,z_2)\psi(u,z_1)\right) = 0, \tag{4.33}$$

where u, v are as in Proposition 4.7. Then for any $U \in \mathcal{Z}_\ell$, e.g., $U = \Omega_W$ for some $W \in \mathcal{C}_\ell$, $\psi(u,z)$ for $u \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$ linearly span a generalized vertex pre-algebra of parafermion operators on U, which by Theorem 3.16 generates a canonical generalized vertex algebra inside $(\mathrm{End}\, U)\{z\}$ with $G = \mathbf{h},\ c(\cdot,\cdot) = 1,\ (\cdot,\cdot) = \frac{1}{\ell}\langle \cdot,\cdot\rangle$.

Lemma 4.8 *Let $U \in \mathcal{Z}_\ell$ and let V be the generalized vertex algebra generated by $\psi(u,z)$ for $u \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$ inside* $(\mathrm{End}\, U)\{z\}$*. Then V is a natural object of $\mathcal{Z}_\ell$ where*

$$h \cdot \phi(z) = [h, \phi(z)], \tag{4.34}$$

$$Z_V(u, z_0) = Y_V(\psi(u,z), z_0) z_0^{\frac{1}{\ell}\alpha(0)} \tag{4.35}$$

for $h \in \mathbf{h},\ \phi(z) \in V,\ u \in \mathfrak{g}_\alpha,\ \alpha \in \Phi$.

Proof. Since U is an $\mathbf{h}$-weight module, End U is a natural $\mathbf{h}$-module where

$$h \cdot f = [h, f] \ (= hf - fh) \quad \text{for } h \in \mathbf{h},\ f \in \text{End } U. \tag{4.36}$$

Then $(\text{End } U)\{z\}$ is a natural $\mathbf{h}$-module. Since the generators $\psi(a, z)$ for $a \in \mathfrak{g}_\alpha$ of V are $\mathbf{h}$-eigenvectors (recall (4.27)), using the proof of Lemma 3.19 we can easily show that V is an $\mathbf{h}$-weight module and (4.27) holds on V. Note that U is a faithful V-module. Then it follows immediately from Lemma 2.8 that (4.29) holds on V. This shows that V is a natural object of $\mathcal{Z}_\ell$. $\square$

Let $\ell \in \mathbf{C}^\times$. Consider the generalized Verma $\hat{\mathfrak{g}}$-module

$$M(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbf{C}[\mathbf{t}]+\mathbf{C}\mathbf{c})} \mathbf{C}_\ell, \tag{4.37}$$

where $\mathbf{C}_\ell = \mathbf{C}$ as a vector space and $\mathfrak{g} \otimes \mathbf{C}[\mathbf{t}]$ acts as zero on $\mathbf{C}_\ell$ and c acts as ℓ. Denote by $\mathbf{1}$ the highest weight vector $1 \otimes 1$ of $M(\ell, 0)$. Let $L(\ell, 0)$ be the (unique) irreducible quotient module with $\mathbf{1}$ as a fixed highest weight vector. Identify $\mathfrak{g}$ as a subspace of $M(\ell, 0)$ and $L(\ell, 0)$ through $a \mapsto a(-1)\mathbf{1}$. Then we have

$$\mathfrak{g}_\alpha \subset \Omega^\alpha_{M(\ell,0)} \subset \Omega_{M(\ell,0)} \quad \text{for } \alpha \in \Phi. \tag{4.38}$$

Theorem 4.9 *Let $\ell \in \mathbf{C}^\times$ and $V = M(\ell, 0)$ or $L(\ell, 0)$. Then there exists a unique generalized vertex algebra structure Y_Ω on Ω_V with $G = \mathbb{Q}$, $c(\cdot, \cdot) = 1$ and $(\cdot, \cdot) = \langle \cdot, \cdot \rangle/\ell$ such that $Y_\Omega(\mathbf{1}, z) = 1$ and $Y_\Omega(a, z) = \psi(a, z)$ for $u \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$. Furthermore, Ω_V is generated by $\mathfrak{g}_\alpha$ ($\alpha \in \Phi$).*

Proof. Clearly, V is $\mathbb{Q}$-graded. Then we take $G = \mathbb{Q}$, $(\cdot, \cdot) = \langle \cdot, \cdot \rangle/\ell$ and $c(\cdot, \cdot) = 1$. Let A be the linear span of $\psi(a, z)$ for $a \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$. It follows from Proposition 4.7 and (4.32) that A is a generalized vertex pre-algebra. Since Ω_V is generated from $\mathbf{1}$ by all the components of $Z(a, z) = \psi(a, z) z^{\alpha(0)/\ell}$, there exists a unique generalized vertex algebra structure Y_Ω on Ω_V with the required conditions. $\square$

Proposition 4.10 *Let $\ell \in \mathbf{C}^\times$ and let $U \in \mathcal{Z}_\ell$. Then U is a natural $\Omega_{M(\ell,0)}$-module. In particular, Ω_W is an $\Omega_{M(\ell,0)}$-module for $W \in \mathcal{C}_\ell$.*

Proof. Set $M = \Omega_{M(\ell,0)} \oplus U$, an object of $\mathcal{Z}_\ell$. Let V be the generalized vertex algebra generated by $\psi(v, z)$ for $v \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$ inside $(\text{End } \Omega_{M(\ell,0)})\{z\} \subset (\text{End } M)\{z\}$. Then M is a V-module with $\Omega_{M(\ell,0)}$ and U as submodules. From Proposition 3.19, there is a V-homomorphism f from V onto $\Omega_{M(\ell,0)}$, which maps $I(z)$ to $\mathbf{1}$. In view of Lemma 4.8, V is a natural $\mathbf{h}$-module in $\mathcal{Z}_\ell$. It follows from Corollary 4.5 that f is a linear isomorphism, hence $V = \Omega_{M(\ell,0)}$. Thus M is an $\Omega_{M(\ell,0)}$-module. Therefore, U is an $\Omega_{M(\ell,0)}$-module. $\square$

We define an $\Omega_{M(\ell,0)}$-$\mathbf{h}$-module to be an $\Omega_{M(\ell,0)}$-module and an $\mathbf{h}$-weight module such that (4.27) holds. In view of Proposition 4.10 and Lemma 4.8 we immediately have:

Corollary 4.11 *The category $\mathcal{Z}_\ell$ is canonically isomorphic to the category of* $\Omega_{M(\ell,0)}$-**h**-*modules.* $\square$

Remark 4.12 Note that for a positive integer ℓ, the generalized vertex operator algebra $\Omega^B_{L(\ell,0)}$ constructed in [DL2] is a quotient algebra of $\Omega_{M(\ell,0)}$. This is recently studied in [Li4] from a different point of view.

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